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Long-time divergence of semiclassical form-factors

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Abstract. The quantum form-factor $K(\tau)$—the Fourier transform of the spectral auto-correlation function—may be represented semiclassically in terms of a sum over classical periodic orbits. We consider the problem of how this approximation behaves in the limit of long (scaled) time τ. It is shown that whilst K itself tends to unity, the periodic-orbit sum typically grows exponentially as $\tau \to \infty$. This behaviour is related to the fact that leading-order semiclassical quantization methods yield complex eigenvalues with imaginary parts that are of higher order in Planck's constant. Divergence from the quantum limit begins when $\tau = \tau^*(\hbar)$, which, for typical two-degrees-of-freedom systems and maps, is shown to be independent of $\hbar$ as $\hbar \to 0$. In the case of the baker's map, however, quantum diffraction from the classical discontinuity instead causes the analogue of τ^* to tend to zero like $N^{-1/2}$, where N is the integer that corresponds to the inverse of Planck's constant. This is in agreement with recent numerical studies. Finally, we consider the implications of the semiclassical divergence studied here for the method developed by Argaman *et al* (1993) of investigating correlations between the periodic orbits of chaotic systems.

1. Introduction

One of the main themes in the study of the quantum properties of classically chaotic systems has been the conjecture that energy-level spectra exhibit universal statistical correlations which are the same as those found in random matrix theory (Berry 1987, Bohigas 1991). This phenomenon is essentially semiclassical in nature and a theory has been developed, based on the leading-order asymptotics of the spectral density of states as $\hbar \to 0$ (Gutzwiller 1971), which explains some features of this universality (and deviations therefrom) in terms of properties of the associated classical motion (Hannay and Ozorio de Almeida 1984, Berry 1985). My purpose in the present paper is to pursue this theory to its limits in order to see where and how it ultimately breaks down.

The analysis of Berry (1985) focussed on the spectral form-factor $K(\tau)$, the Fourier transform of the eigenvalue pair correlation function, and provided an explicit expression for $K^{\rm sc}(\tau)$, the leading-order semiclassical approximation to K, in terms of a sum involving the set of classical periodic orbits. Applying a sum rule derived in Hannay and Ozorio de Almeida (1984), this could then be estimated for values of τ (the time measured in units of $2\pi\hbar\bar{d}$, where $\bar{d}$ is the mean spectral density) in the range $\tau \ll 1$. The results agreed exactly with the corresponding limit of the appropriate Random Matrix Theory form-factors. In the region $\tau \gg 1$, the limiting behaviour was probed using a semiclassical sum-rule for the trace formula, obtained by appealing to properties of the fully quantum density of states function. The conclusion was that for all systems with a non-systematically degenerate spectrum, $K^{\rm sc}(\tau) \approx 1$ in this range. Essentially, the argument involved is equivalent to the assumption that since it can be shown that $K(\tau) \approx 1$, then $K^{\rm sc}$ should exhibit the same

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behaviour. Again, the result agrees with the corresponding limit of the associated random matrix theory statistics.

It is this second stage in the semiclassical analysis which will be studied in more detail here. Our aim is the see how the long-time behaviour of K^{sc} is affected by the fact that it represents only a leading-order asymptotic result, that is, whether higher-order terms in the semiclassical expansion can indeed be neglected without compromising the ability of K^{sc} to approximate K in the range $\tau \gg 1$, where it is most sensitive to the detailed properties of the eigenvalue spectrum. In fact we shall find that for typical two-degree-of-freedom systems and for phase space maps they cannot. Specifically, the arguments to be presented here lead to the conclusion that the sum over periodic orbits which defines K^{sc} diverges exponentially as $\tau \to \infty$.

The method we use to show this is based on the fact that leading-order semiclassical quantization formulae typically give rise to energy-level approximations that are complex. The actual levels themselves must, of course, be real and so this might seem at first sight to be to be something of a paradox. The explanation is that the imaginary parts, which arise due to the approximations involved, are of higher order in powers of $\hbar$ than the real parts. It means that K^{sc}, the expression for which was derived using only the leading-order result, in fact contains statistical information about complex eigenvalues, and it is the imaginary parts of these that give rise to a divergence in the long-time limit. (Our results thus have no connection with those of Aurich and Sieber (1993), which concern a class of systems for which Gutzwiller's formula is exact.)

There are two reasons for our interest in the behaviour of K^{sc} in the range $\tau \gg 1$. First, there have recently been a number of important developments in the general study of the long-time properties of semiclassical formulae, especially with regard to the propagation of wavefunctions beyond the 'chaos time' (Heller and Tomsovic 1993). Our results are complementary to these in the sense that they are concerned with time-scales that are much longer, being of the order of $\hbar^{-1}$ for two-degree-of-freedom systems and maps as opposed to $\log(\hbar^{-1})$. Second, the semiclassical approximation to the quantum form-factor was recently employed as the basis of a method to investigate correlations between the actions of the periodic orbits in strongly chaotic systems (Argaman *et al* 1993). This involved the key assumption that K can be approximated by K^{sc} over a suitably large range of values of τ. Restrictions on this range limit the information that can be obtained about the orbit correlations. In connection with this work, a numerical evaluation of K^{sc} was carried out for the baker's map (Argaman *et al* 1993, Dittes *et al* 1993), which showed, surprisingly, an exponential growth beginning at a time $\tau^*(\hbar)$ such that $\tau^*(\hbar) \to 0$ in the semiclassical limit. This is clearly at variance with the known behaviour of K itself. Obviously, if it represents a general phenomenon, this casts serious doubt on the validity of the action-correlation analysis. We shall argue that it does not. A divergence is to be expected for typical systems, but not of the form found for the baker's map. In particular, it typically begins at values of τ large enough for it not to have such serious implications for the results derived by Argaman *et al*. The properties of the baker's map are thus non-generic in this respect and are traceable to the discontinuous nature of the transformation.

2. The semiclassical form-factor

The form-factor $K(\tau)$ is defined to be the Fourier transform of the spectral autocorrelation function. Thus if the density of states for the set of eigenvalues E_n,

$$d(E) = \sum_n \delta(E - E_n) \tag{1}$$

is split into its mean $\bar{d}$ (given by the Weyl series) and fluctuations thereabout $\tilde{d}$, then

$$K(\tau) = \frac{1}{\bar{d}^2} \int_{-\infty}^{\infty} \left\langle \tilde{d}\left(E + \frac{\varepsilon}{2\bar{d}}\right) \tilde{d}\left(E - \frac{\varepsilon}{2\bar{d}}\right) \right\rangle_E \exp(2\pi i \varepsilon \tau)\, d\varepsilon \tag{2}$$

where the angular brackets denote an average with respect to E over a range ΔE such that $\bar{d}^{-1} \ll \Delta E \ll E$, i.e.

$$\langle f(E) \rangle_E \equiv \frac{1}{\Delta E} \int_{E-\Delta E/2}^{E+\Delta E/2} f(E')\, dE'. \tag{3}$$

It is clear from its definition that K is an even function of τ.

A semiclassical approximation to K may be obtained by substituting for $\tilde{d}$ its leading-order asymptotic form as $\hbar \to 0$:

$$\tilde{d}(E) \approx \frac{1}{\pi\hbar} \operatorname{Re} \sum_j A_j \exp\left(iS_j/\hbar - i\pi\upsilon_j/2\right) \tag{4}$$

where $S_j(E)$ and υ_j represent the action and Maslov index of the jth periodic orbit of the corresponding classical flow, and $A_j(E)$ is related to both the associated stability and to the period $T_j(E)$ (Gutzwiller 1971). The result (Berry 1985) is that

$$K(\tau) \approx K^{sc}(\tau) \tag{5}$$

where

$$K^{sc}(\tau) \equiv \frac{1}{2\pi\hbar\bar{d}} \left\langle \sum_i \sum_j A_i A_j \exp\left\{\frac{1}{\hbar}\left(S_i - S_j\right) - \frac{i\pi}{2}\left(\upsilon_i - \upsilon_j\right)\right\} \delta\left(T - \frac{T_i + T_j}{2}\right) \right\rangle_E \tag{6}$$

with $T = 2\pi\hbar\bar{d}\tau$. It is important for our purposes to note that because the periodic orbit sum (4) typically does not converge in the region of integration in (2), this expression for K^{sc} is essentially formal. In fact, convergence is only achieved, and hence the above substitution is only valid, provided the energy E is given a sufficiently large imaginary part (Eckhardt and Aurell 1989). We shall return to this point again later since it is central to the main problem of interest here. First we will explain precisely what this problem is.

The general question that we intend to focus on is the following: over which range of values of τ does the approximation (5) hold? In particular, is the double sum in (6) able to reproduce the limiting behaviour $K(\tau) \to 1$ as $\tau \to \infty$? In order to answer this, we shall begin by first outlining an argument which leads to the quantum result itself. Our analysis of the corresponding behaviour of K^{sc} will then follow similar lines.

The long-time limit of $K(\tau)$ can be extracted directly from an explicit expression which derives from the definitions (1), (2) and (3), together with the decomposition $d = \bar{d} + \tilde{d}$:

$$K(\tau) = \frac{1}{\Delta E \bar{d}} \sum_{|E - \frac{E_n + E_m}{2}| \leqslant \frac{\Delta E}{2}} \exp\left\{2\pi i \bar{d}\tau (E_n - E_m)\right\} - \delta(0) \tag{7}$$

where the sum includes all pairs of eigenvalues whose mean lies within a range of size ΔE, centred on E. In the limit as $\tau \to \infty$, all of the terms in the sum are increasingly oscillatory, except for those corresponding to pairs such that $E_n = E_m$. Assuming no systematic degeneracies, there are approximately $\Delta E\bar{d}(E)$ of these constant terms in the range of summation. Hence, after any amount of smoothing it is clear that $K(\tau) \to 1$. Also apparent is the fact that this limiting behaviour emerges when $\bar{d}\tau\overline{|E_{n+1} - E_n|} \gg 1$, that is, when $\tau \gg 1$.

The large-τ behaviour of K^{sc} was first considered in Berry (1985). The approach taken there was based on the semiclassical sum rule that follows directly from the assumption that the quantum identity

$$\lim_{\varepsilon \to 0} 2\pi\varepsilon d^2(E + i\varepsilon) = d(E) \tag{8}$$

which holds for non-degenerate spectra is satisfied by the semiclassical periodic orbit approximation to the density of states as well. It was shown that if this is valid then it implies that K^{sc} has the same limiting form as K. Our aim here is to investigate whether this is indeed the case by deriving a semiclassical analogue of (7) and then applying an argument closely related to that used above for K itself.

The starting point of our analysis is the extension of (4) into the complex energy plane. This is necessary because, as noted earlier, the periodic orbit sum does not converge when E is real, that is, where the integration in (2) is to be carried out (for classically integrable systems it converges when E is given any non-zero imaginary part, whereas for classically chaotic systems one must go a finite distance into the complex E plane, i.e. beyond the 'entropy barrier'—the difference between the topological entropy and half the metric entropy). The appropriate result is that

$$\tilde{d}(E + i\alpha) \approx \frac{1}{\pi\hbar} \operatorname{Re} \sum_j A_j \exp\left(iS_j/\hbar - i\pi\upsilon_j/2 - \alpha T_j/\hbar\right) \tag{9}$$

which may also be written in the general form

$$\tilde{d}(E + i\alpha) \approx -\frac{1}{\pi} \operatorname{Im} \frac{d}{dE} \log Z(E + i\alpha) \tag{10}$$

where for two-degree-of-freedom systems

$$Z(E) \equiv \prod_p \prod_{m=0}^{\infty} \left[1 - \exp\left\{-\left(m + \tfrac{1}{2}\right)\lambda_p T_p + iS_p/\hbar - i\pi\upsilon_p/2\right\}\right] \tag{11}$$

is the semiclassical zeta-function, the product running over primitive periodic orbits, labelled p, with stability exponents λ_p. The non-trivial zeros of Z are taken to define the semiclassical eigenvalues (i.e. they represent leading order approximations to the exact quantum eigenvalues in the limit as $\hbar \to 0$). In general they are complex and so may be written in the form $E_n^{sc} = x_n + iy_n$. Assuming that Z can be expressed as a product over these zeros it then follows that

$$d(E + i\alpha) = \frac{1}{\pi} \sum_n \frac{\alpha - y_n}{(E - x_n)^2 + (\alpha - y_n)^2}. \tag{12}$$

Equations (9) and (12) represent equivalent semiclassical expressions for $\tilde{d}$ (in the second case, after the subtraction of $\bar{d}$), which we now use to obtain two corresponding formulae for the form-factor.

Taking α to be large enough so that the sum in (9) converges absolutely and uniformly (i.e., focussing for the moment on what happens beyond the entropy barrier), the periodic orbit representation may be substituted directly into (2) to give

$$K_{\alpha}^{sc}(\tau) \equiv \frac{1}{2\pi\hbar\bar{d}}\left\langle \sum_i \sum_j A_i A_j \exp\left\{\frac{1}{\hbar}\left(S_i - S_j\right) - \frac{i\pi}{2}\left(\upsilon_i - \upsilon_j\right)\right\} \times \exp\left\{-4\pi\tau\alpha\bar{d}\right\}\delta\left(T - \frac{T_i + T_j}{2}\right)\right\rangle_E . \tag{13}$$

Alternatively, substituting (12) into (2) leads to the semiclassically equivalent result

$$K_{\alpha}^{sc}(\tau) = \frac{1}{\Delta E\bar{d}} \sum_{\left|E - \frac{x_n + x_m}{2}\right| \leqslant \frac{\Delta E}{2}} \exp\left[2\pi i\bar{d}\tau\left(x_n - x_m\right)\right]\exp\left[2\pi\left(y_n + y_m - 2\alpha\right)\tau\bar{d}\right] - \delta(0) \tag{14}$$

here given in the form appropriate when $\tau > 0$ (the fact that K_{α}^{sc} is an even function of τ immediately leads to the corresponding form when $\tau < 0$). It is important to note that in the derivation of this second expression it has been assumed that $\alpha > y_n \forall n$. This is certainly true when the condition on α noted above is satisfied, because the Euler product (11) has no zeros beyond the entropy barrier where it converges. It is also worth noting that in the case when $\alpha < y_n$ the contribution from the corresponding pole of (12) to the integrals in (2) takes a different form to that given in (14).

Even though the derivation of the periodic orbit expression (13) is only justified when α lies in the region where (9) converges, formally setting $\alpha = 0$ gives the earlier expression (6). Clearly if the same is done in the equivalent formula (14) it follows that

$$K^{sc}(\tau) = \frac{1}{\Delta E\bar{d}} \sum_{\left|E - \frac{x_n + x_m}{2}\right| \leqslant \frac{\Delta E}{2}} \exp\left[2\pi i\bar{d}\tau\left(x_n - x_m\right)\right]\exp\left[2\pi\left(y_n + y_m\right)\tau\bar{d}\right] - \delta(0) \tag{15}$$

which corresponds to the main result of this section. It immediately implies that if any of the semiclassical eigenvalues has a positive imaginary part ($y_n > 0$), then $K^{sc}(\tau)$, as given by (6), will diverge exponentially as τ increases. Specifically, if the largest value of y_n is y then as $\tau \to \infty$

$$K^{sc}(\tau) \sim \frac{1}{\Delta E\bar{d}}\exp\left(4\pi y\tau\bar{d}\right) \tag{16}$$

this exponential increase (or 'lift-off') beginning at $\tau^* \sim \left(y\bar{d}\right)^{-1}$. Conversely, if all of the y_n are negative then the form-factor will decrease exponentially after this value of τ. Hence only when it is the case that $y_n \leqslant 0 \forall n$ and, moreover, $y_n \neq 0$ for a set of states of measure zero will $K^{sc} \to 1$ as $\tau \to \infty$.

The question of how the semiclassical approximation to the form-factor behaves in the long-time limit thus rests on the nature of the imaginary parts y_n of the semiclassical energy levels. These occur because the zeta-function (11), whose zeros represent the levels, is only a leading order asymptotic result as $\hbar \to 0$. Thus, whilst the complete semiclassical expansion for the quantum analogue of Z must have real zeros, this is not the case for the individual terms therein (Keating 1992) and consequently $y_n \neq 0$ for typical states. Since the main contribution to these imaginary parts comes from the second order in the semiclassical expansion, it is obviously expected that $y_n = O(\hbar^2)$ (see, for example, Boasman 1994). Furthermore, there is no reason to believe that all of the y_n have the same sign. Indeed, in a number of the systems that have been studied, they have been found not to (see, for example, Dahlqvist 1992 and Chritiansen and Cvitanovic 1992). Hence it is natural to conjecture that the y_n will typically be distributed around zero (not necessarily symetrically) with a variance that is $O(\hbar^2)$.

The implication of these arguments is that the periodic orbit formula (6) for $K^{sc}(\tau)$ is generically expected to grow exponentially for $\tau > \tau^*$ where $\tau^* = O(\hbar^{-2}\bar{d}^{-1})$, the exponential rate being of the same order. For two-degree-of-freedom systems this gives $\tau^* = O(1)$, that is, we anticipate that K^{sc} will be a good approximation to K up to some τ which is independent of $\hbar$, after which it should diverge as exp(const τ). Moreover, in the numerical investigations noted above it was found that $y_n\bar{d} < 1$, and there is partial support for this too from the result (Boasman 1994) that $|E_n - x_n|$ (which one might expect to be typically about the same size as y_n) is, on average, roughly equal to $0.03\bar{d}^{-1}$. If this is indeed the case then it implies that $\tau^* > 1$, and so the saturation of the semiclassical form-factor may still occur. In a higher dimension D, the situation is worse in that $\tau^* = O(\hbar^{D-2})$ and so the lift-off begins semiclassically close to the origin.

The question now remains as to how the divergence described above arises. Clearly if α is sufficiently large so that the substitutions leading to the periodic orbit sum (13) are valid, then, since this necessarily implies $\alpha > y_n$ in the equivalent form (14), it follows that $K_\alpha^{sc}(\tau) \to 0$ as $\tau \to \infty$. Equation (6) corresponds to formally setting $\alpha = 0$ in (13), that is, it relies on the assumption that this expression for K_α^{sc} is also valid in the region where the trace formula (9) is at best only conditionally convergent. Hence the essential point is whether K^{sc} is indeed the correct limit of K_α^{sc} as $\alpha \to 0$. A comparison with the behaviour of the equivalent expression (14) suggests that in general it is not. In this case it may be verified that as $\alpha \to 0$ the poles of (12) that correspond to semiclassical levels with $y_n > 0$ pass through the contour of integration in (2). Then, as noted above, their contribution to the final result changes. Hence, simply setting $\alpha = 0$ in (14) does not give the correct continuation as $\alpha \to 0$, and this obviously implies the same fate for equation (13).

3. Quantum maps

The analysis of the last section applies directly to the semiclassical form-factors for quantum maps as well. Given, for example, a canonical mapping on the unit 2-torus, here taken to be a two-dimensional phase space, the corresponding quantum propagator is an $N \times N$ unitary matrix U, N playing the role of the inverse of Planck's constant. It follows from the unitarity of U that its eigenvalues lie on the unit circle in the complex plane, their arguments θ_n being known as quasi-energies. The density of states is then defined by

$$d(\theta) = \sum_{n=1}^{N} \sum_{k=-\infty}^{\infty} \delta(\theta - 2\pi k - \theta_n) \tag{17}$$

and has the mean value $N/2\pi$. A form-factor may be constructed, by analogy with (2), in terms of the fluctuations $\tilde{d} \equiv d - N/2\pi$ about this value:

$$K(L) = \frac{2\pi}{N^2} \int_0^{2\pi} \mathrm{d}\theta \int_0^N \mathrm{d}\varepsilon\, \tilde{d}\left(\theta + \frac{\pi\varepsilon}{N}\right) \tilde{d}\left(\theta - \frac{\pi\varepsilon}{N}\right) \exp\left(\frac{2\pi \mathrm{i}\varepsilon L}{N}\right). \tag{18}$$

As for flows, our primary interest is in the long-time limit $L \to \infty$.

Substituting (17) into (18) leads directly to the explicit formula

$$K(L) = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \exp\left[\mathrm{i}L(\theta_n - \theta_m)\right] - N\delta_{L,0} \tag{19}$$

where $\delta_{i,j}$ is the Kronecker δ-symbol. It follows that as $L \to \infty$, any amount of smoothing leaves $K(L) \sim 1$, this behaviour emerging when $L\overline{|\theta_n - \theta_m|} \gg 2\pi$, i.e. when $L \gg N$. The expression (19) is, by definition, equivalent to

$$K(L) = \frac{1}{N} \left|\mathrm{Tr}\, U^L\right|^2 - N\delta_{L,0} \tag{20}$$

and a semiclassical approximation to K, K^{sc}, may be obtained by substituting for $\mathrm{Tr}\, U^L$ its leading-order asymptotic representation: as $N \to \infty$,

$$\mathrm{Tr}\, U^L \approx \sum_j B_j \exp\left(2\pi \mathrm{i} N S_j - \mathrm{i}\pi\upsilon_j/2\right) \tag{21}$$

where j labels fixed points of the Lth power of the classical map, with S_j, B_j and υ_j their action, stability amplitude and Maslov index respectively (Tabor 1983). The question analogous to that discussed in the previous section is: does this approximation accurately describe the behaviour of K in the long-time limit?

To answer this, we proceed exactly as in the case for flows, and so omit some of the details. The semiclassical approximation to the fluctuating part of the density of states may be written in terms of a zeta-function, defined in analogy with the Euler product (11) and whose non-trivial zeros we take to represent the semiclassical quasi-energies $\theta_n^{\mathrm{sc}} \equiv \phi_n + \mathrm{i}\eta_n$. In the complex θ plane, at a point with imaginary part α that is sufficiently large for the product to converge (i.e. beyond the entropy barrier), this implies that

$$d(\theta + \mathrm{i}\alpha) \approx -\frac{1}{\pi} \mathrm{Im} \sum_{n=0}^{N-1} \sum_{k=-\infty}^{\infty} \frac{1}{\theta + \mathrm{i}\alpha - 2\pi k - \phi_n - \mathrm{i}\eta_n}. \tag{22}$$

It follows from the condition on the size of α that $\eta_n < \alpha\, \forall n$. Evaluating the sum over k in this case using the Poisson summation formula then gives

$$\tilde{d}(\theta + \mathrm{i}\alpha) \approx \frac{1}{\pi} \mathrm{Re} \sum_{j=1}^{\infty} \exp(ij\theta - \alpha j) \sum_{n=0}^{N-1} \exp\left[-ij(\phi_n + \mathrm{i}\eta_n)\right] \tag{23}$$

and substituting this expression into (18) leads finally to

$$K_\alpha^{\mathrm{sc}}(L) = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \exp\left[\mathrm{i}L(\phi_n - \phi_m) + L(\eta_n + \eta_m - 2\alpha)\right] - N\delta_{L,0} \tag{24}$$

when $L \geqslant 0$ (the result for $L < 0$ following immediately from the fact that K_α^{sc} is, by definition, an even function of L).

Formally setting $\alpha = 0$ in (24) gives the result that corresponds to substituting the fixed-point sum (21) into (20). Hence it may be seen that if any of the semiclassical eigenvalues has a positive imaginary part, then K^{sc} diverges exponentially as $L \to \infty$, this behaviour emerging when $L > \eta^{-1}$, where η represents the largest η_n. If, on the other hand, the imaginary parts are all negative, then K^{sc} decreases exponentially. Only when $\eta_n \leqslant 0$ and the fraction of states with $\eta_n \neq 0$ tends to zero as $N \to \infty$ does the semiclassical approximation fully reflect the long-time limit $K \to 1$. As in the case of flows, there is, however, no reason to expect this to be the case. In fact, there is also no reason to believe that the η_n cannot be positive, and typically one would expect one or more to be so. Hence it is again natural to conjecture that long-time exponential divergence is the generic behaviour.

As also is the case for flows, the imaginary parts of the semiclassical eigenvalues arise from corrections to the asymptotic approximation (21). Hence they should typically be of the order of the next term in the full semiclassical expansion, i.e. $\eta_n = O(1/N)$. This means that the divergence should begin at $L^* \sim N$, or, in terms of the scaled variable corresponding to τ, $l = L/N$, at $l^* \sim O(1)$ with respect to N. The results of Boasman (1994) may also be taken to imply that $l^* > 1$, which indeed appears to be the case for the perturbed cat map studied in Argaman *et al* (1993).

In the case of the baker's map (Balazs and Voros 1988), however, a different behaviour is expected. Here corrections to the fixed point sum (21) are not due solely to the later terms in a stationary phase expansion; they also include contributions related to diffraction from the discontinuity of the classical transformation. This is particularly clear from the optical realization of the map and its quantization given by Hannay *et al* (1994). There the discontinuity was shown to be equivalent to that arising at the edge of a finite wedge (essentially, because the map is linear on either side of it). Hence, applying the Sommerfeld solution (see, for example, Jones 1964), leads to the conclusion that the corrections should be $O(1/\sqrt{N})$. (Physically, this result follows from the fact that the diffracted rays may be viewed as coming from a point source—the discontinuity—and so in two dimensions the ratio of the amplitude of the associated field to that representing the non-diffracted rays is $O(\sqrt{\text{wavelength}})$, which is equivalent to $O(\sqrt{\hbar})$, that is, to $O(1/\sqrt{N})$ for maps.)

An explicit confirmation of this behaviour follows from the recent semiclassical studies of Saraceno and Voros (1993), and, in particular, of da Luz and Ozorio de Almeida (1994). Their results imply that for the baker's map the next correction to (21) associated with a given periodic orbit takes the form of a sum of terms, each corresponding to when a point on the orbit lies within a distance $1/N$ from the edge of the unit square which represents the phase space. For an orbit of period n, ergodicity implies that the number of these contributions will be $\sim n/N$ as $n \to \infty$. However, they come with phases that are approximately random, and so the overall order is $\sqrt{n/N}$. Hence for fixed n, the semiclassical corrections are $O(1/\sqrt{N})$, as deduced above.

This atypical behaviour of the quantum baker's map obviously has important consequences for the problem under discussion. Clearly it leads to the expectation that $\eta_n = O(1/\sqrt{N})$, i.e. that the imaginary parts of the semiclassical eigenvalues are $O(\sqrt{N})$ larger than the mean separation. Hence the associated exponential divergence of the form-factor will begin at $L^* \sim \sqrt{N}$, and the growth will be of the form $\exp(\alpha L/\sqrt{N})$. In the scaled variable $l = L/N$, it therefore begins at $l^* \sim 1/\sqrt{N}$. This is precisely the behaviour found to hold in recent numerical studies (Argaman *et al* 1993, Dittes *et al* 1993). It means that as $N \to \infty$, the divergence begins closer and closer to $l = 0$ and so ultimately it destroys the initial agreement with the random matrix theory COE result.

Finally, returning to maps in general, it may be noted that, as for flows, the reason for the appearance of a divergence is that the result of simply substituting (21) directly into (20) does not actually correspond to the correct continuation of the semiclassical form-factor from the region where the periodic orbit sum converges. This is clear from the derivation of the equivalent expression (24) from (22) via (23), which holds only when it is the case that $\alpha > \eta_n \, \forall n$. As $\alpha \to 0$, poles corresponding to semiclassical levels with $\eta_n > 0$ pass through the contour of integration and so have to be treated separately, giving rise to contributions which take a different form to that represented in (24). Hence, simply setting $\alpha = 0$ (which gives $K^{\rm sc}$) does not actually correspond to the correct limit of $K^{\rm sc}_\alpha$ as $\alpha \to 0$ from the region of convergence unless $\eta_n \leqslant 0$.

4. Periodic orbit correlations

Based on the assumption that the semiclassical form-factors of typical systems are well approximated by the corresponding random matrix ensemble statistics, it was recently shown that the periodic orbits of chaotic systems may be expected to exhibit non-trivial universal correlations (Argaman *et al* 1993). The orbit correlation function considered was

$$P(x;T) \equiv \Delta T \sum_{i \neq j} A_i A_j (-1)^{\upsilon_i - \upsilon_j} \delta\left(x - \left[S_i - S_j\right]\right) \delta_{\Delta T}(T - T_i)\, \delta_{\Delta T}\left(T - T_j\right) \tag{25}$$

where $\delta_{\Delta T}$ is a δ-function of width ΔT and the sum includes all pairs of non-identical periodic orbits. (The form given in equation (25) is actually the one appropriate for cases when the υ_i are all even). P is closely related to the Fourier transform of the form-factor $K\left(T/2\pi\hbar\bar{d}\right)$ in that if

$$\hat{P}(y) \equiv \frac{V}{2\pi T^2} P\left(\frac{Vy}{2\pi T}; T\right) \tag{26}$$

$V(E)$ being the volume of the surface of constant energy E in phase space, then

$$\hat{P}(y) \approx \frac{1}{\pi} \int_0^\infty \left(K^{\rm sc}(1/z) - g/z\right) z \cos(zy)\, {\rm d}z \tag{27}$$

where g is a symmetry-dependent constant such that g/z represents the contribution to $K^{\rm sc}(1/z)$ from terms in the double sum (6) in which i and j label the same orbit. An explicit form for $\hat{P}$ was derived in Argaman *et al* by evaluating the integral in equation (27) under the assumptions that $K^{\rm sc} \approx K$ and that K itself is given by the appropriate random matrix form-factor.

It may now be seen that the arguments outlined in sections 2 and 3 imply that the first of these approximations has only a limited range of validity, since the relationship between $K(1/z)$ and $K^{\rm sc}(1/z)$ is expected to break down in the limit as $z \to 0$. This consequently restricts the range of values of y for which $\hat{p}$ can be calculated in this way. Specifically, if, as expected for typical two-degree of freedom systems and maps, $K^{\rm sc}$ diverges when $1/z > \tau^* = O(1)$, then the results obtained from (27) for $\hat{P}$ are necessarily limited to the range $y < y^* = O(1)$. Fortunately, this is the region of most physical interest, since the action repulsion uncovered for chaotic systems appears as $y \to 0$.

The atypical behaviour of the baker's map with respect to the long-time limit of its semiclassical form-factor would appear to imply that the method of deriving action correlations described above should not really be applicable in that case. It is therefore extremely interesting that the correlations predicted were, nevertheless, still found amongst its periodic orbits. This suggests that in general their existence may persist beyond the limit

y^*; that is, the semiclassical divergence might influence only the method of derivation and not the actual form of $\hat{P}$.

There are two more reasons for believing that the action correlations might be more robust than is suggested by the divergence-related limitation on the above method of deriving them. The first is that it is important to note that the long-time behaviour discussed here is directly related to the fact that the periodic orbit formulae (9) and (21) are in general semiclassical approximations, and that therefore they only reflect the Hermitian structure of quantum mechanics to leading order as $\hbar \to 0$. Hence for systems where these expressions are in fact exact, such as compact billiards on surfaces of constant negative curvature (Balazs and Voros 1986) and the cat maps (Keating 1991), no such divergence will occur, because the zeros of their zeta functions are identically equal to the quantum eigenvalues and so must be real. (For Neumann boundary conditions, the divergence discovered by Aurich and Sieber (1993) may, of course, still occur, however it appears that this too is due to a complex pole which restricts the continuation of the form-factor from the region where the manipulations leading to the analogue of (13) are legitimate.) Hence in these systems there is no fundamental difficulty in applying the method described above and hence of obtaining the action correlation function. But these examples are generally viewed as being paradigms of chaotic dynamics and so the presence of correlations amongst their periodic orbits points to the wider existence of the phenomenon. Furthermore, since a number of them have also been shown to possess GOE random matrix level-statistics, this also provides explicit support for the universality of the action correlation result.

The second reason for believing that the existence of these correlations may transcend the problems caused by the long-time divergence of the form-factor is that in principle the definition of the correlation function (25) could be extended to reflect additional contributions to the semiclassical approximation to K from higher order corrections to (9) and (21) (such as those discussed by Gaspard and Alonso 1993). But it is clear from the arguments presented in the previous sections that the inclusion of these additional terms would mean that the imaginary parts of the energy levels would be of higher order in Planck's constant and, as a consequence, that the divergence would be postponed to a point $\tau^{**}(\hbar)$ such that $\tau^{**}(\hbar) \to \infty$ as $\hbar \to 0$. The method of Argaman *et al* (1993) could then be applied directly to yield the same universal forms for a slightly modified correlation function.

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